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Isotropic random flights: random numbers of flights

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Abstract. Isotropic random flights, where the number of individual flights N is random, are studied. N is taken to be governed by a Poisson distribution and also by a negative binomial distribution, each with mean $\langle N \rangle$. The probability density function of the length of the vector sum is shown to be mixed, in that it contains impulse components (Dirac delta functions) as well as the absolutely continuous component. The limiting density functions are also obtained, and in the negative binomial case lead to the random flight version of the K-density function introduced by Jakeman and collaborators. Finally, the moments about the origin are explicitly evaluated for both fixed N and random N.

1. Introduction

The problem of isotropic random flights arises in a variety of physical and technical areas. Rayleigh (1919), Chandrasekhar (1943) and Flory (1969) have summarised the formal development of the analysis leading to the probability density function (PDF) of the length of the resultant vector sum in terms of an infinite integral with an oscillatory integrand. When the lengths of the flights are equal, the integral can be evaluated explicitly in terms of polynomials with a different polynomial in each subinterval between discontinuities in slope; Treloar (1946) and Vincenz and Bruckshaw (1960) have independently determined the polynomials. A purely numerical approach to the evaluation of the integral was advocated by Jernigan and Flory (1969). Barakat (1973) considered the more general situation where the length of each flight is random and employed the sampling expansion to evaluate the integral.

The purpose of the present paper is to extend the analysis to the situation where the number of flights, N, is also allowed to be random. Applications of the results to some physical problems (in solid state) are the subject of a second paper in preparation. The discrete random variable N is taken to obey a Poisson distribution, and a negative binomial distribution. The negative binomial distribution often gives an adequate representation when the strict randomness requirements for the Poisson distribution are not approximated sufficiently closely. The resultant radial density functions are shown to be mixed because they contain impulse (Dirac delta function) components in addition to the absolutely continuous components. The limiting form of the radial density functions as the average number of flights is increased is also examined; in the negative binomial case it is shown that the limiting form is a K-density function. The first four moments about the origin are also explicitly evaluated for both fixed and random N for the general case where the individual flights are random, and their asymptotic behaviour is investigated. Finally, some remarks pertinent to infinitely divisible density functions and to stable density functions in the context of the random flight problem are discussed.

Such generalisations of the corresponding two-dimensional walk problem have already been studied with regard to problems in light scattering, photoelectron counting, laser speckle patterns, multipath propagation, etc. Some references are: Barakat (1981), Barakat and Blake (1976), Barakat and Cole (1979), Chen and Tartaglia (1972), Chrostowski and Zardecki (1978), Hoenders *et al* (1979), Jakeman (1980a, b, containing extensive references), Jakeman and Pusey (1973, 1976, 1978), Pusey (1977), Schaefer and Pusey (1972).

2. Deterministic N

We seek the PDF of the length $R \equiv |\boldsymbol{R}|$ of the vector sum

$$\boldsymbol{R} = \sum_{n=1}^{N} \boldsymbol{r}_n \tag{2.1}$$

and call it $f(R|N)$. The $\boldsymbol{r}_n$ are statistically independent vectors possessing isotropic PDFs

$$w(\boldsymbol{r}_n) = (4\pi^2)^{-1} w(r_n), \qquad r_n \equiv |\boldsymbol{r}_n|. \tag{2.2}$$

The corresponding characteristic function of r is the three-dimensional Fourier transform of $w(r_n)$, which because of isotropy becomes

$$A(\rho_n) = \int_0^\infty r_n^2 w(r_n)\left(\frac{\sin \rho_n r_n}{\rho_n r_n}\right) \mathrm{d}r_n, \tag{2.3}$$

where $\rho_n \equiv |\boldsymbol{\rho}_n|$. The characteristic function of their sum, $A(\rho|N)$, is the product of the individual characteristic functions because the individual vectors are statistically independent. We are interested in the case where all vectors $\boldsymbol{r}_n$ have the same density function

$$A(\rho|N) = \left[\int_0^\infty r^2 w(r)\left(\frac{\sin \rho r}{\rho r}\right) \mathrm{d}r\right]^N = [A(\rho|1)]^N. \tag{2.4}$$

As shown in Chandrasekhar (1943), $f(R|N)$ is given by

$$f(R|N) = \frac{2R}{\pi}\int_0^\infty A(\rho|N)\rho \sin R\rho \, \mathrm{d}\rho. \tag{2.5}$$

When N is large, the characteristic function behaves as a Gaussian in the vicinity of the origin,

$$A(\rho|N) \sim \exp(-q^2\rho^2), \tag{2.6}$$

where

$$q \equiv (N\langle r^2\rangle/6)^{1/2}. \tag{2.7}$$

The corresponding density function is Maxwellian,

$$f(R|N) = (2\pi)^{-1/2} q^{-3} R^2 \exp(-R^2/4q^2). \tag{2.8}$$

If all the flights are of equal fixed length l

$$w(r) = r^{-2}\delta(r - l) \tag{2.9}$$

and

$$\langle r^{2k}\rangle = \int_0^\infty r^{2k+2} w(r)\,\mathrm{d}r = l^{2k}. \tag{2.10}$$

For flights obeying a rectangular density function

$$\begin{aligned} w(r) &= 1/r^2, \qquad 0 \leqslant r \leqslant l, \\ &= 0, \qquad \text{elsewhere}, \end{aligned} \tag{2.11}$$

it follows that

$$\langle r^{2k}\rangle = l^{2k}/(2k+1). \tag{2.12}$$

The respective q's are:

(a) Dirac density function $\quad q = (Nl^2/6)^{1/2}$,

(b) rectangular density function $\quad q = (Nl^2/18)^{1/2}$.

The sampling theorem has been used to evaluate $f(R|N)$ in terms of sampled values of $A(\rho|N)$, when $A(\rho|N)$ is a band-limited function (Barakat 1973), in place of direct quadrature of equation (2.5). The expression is

$$\begin{aligned} f(R|N) &= 2\left(\frac{R}{B}\right) \sum_{m=1}^{\infty} \left(\frac{m\pi}{B}\right) A\left(\frac{m\pi}{B}\Big|N\right) \sin\left(\frac{m\pi R}{B}\right), \qquad 0 \leqslant R \leqslant B, \\ &= 0, \qquad \text{elsewhere}. \end{aligned} \tag{2.13}$$

Here, $B \equiv N\beta$, where β is that value of r for which $w(r) \equiv 0$ if $r > \beta$. The smoother $f(R|N)$, the more rapid is the convergence of its Fourier series expansion. Reference is made to the original paper for the derivation of equation (2.13) and representative numerical results.

The radial distribution function $F(R'|N)$, which is the probability that $R' > B$, is

$$F(R'|N) = \int_0^{R'} f(R|N)\,\mathrm{d}R. \tag{2.14}$$

$F(R'|N)$ can also be expressed as a Fourier series

$$\begin{aligned} F(R'|N) &= \frac{(2\pi)^{1/2} R'}{B} \sum_{m=1}^{\infty} \left(\frac{m\pi R'}{B}\right)^{1/2} A\left(\frac{m\pi}{B}\Big|N\right) J_{3/2}\left(\frac{m\pi R'}{B}\right), \qquad R' < B, \\ &= 1, \qquad R' > B. \end{aligned} \tag{2.15}$$

The moments about the origin $\langle R^{2k}|N\rangle$, $k = 0, 1, 2, \ldots$, can be evaluated by differentiation of the characteristic function, equation (2.4). The first four moments are:

$$\begin{aligned} \langle R^2|N\rangle &= N\langle r^2\rangle, \\ \langle R^4|N\rangle &= N\langle r^4\rangle + \tfrac{5}{3}N(N-1)\langle r^2\rangle^2, \\ \langle R^6|N\rangle &= N\langle r^6\rangle + 7N(N-1)\langle r^2\rangle\langle r^4\rangle + \tfrac{35}{9}N(N-1)(N-2)\langle r^2\rangle^3, \\ \langle R^8|N\rangle &= N\langle r^8\rangle + N(N-1)[\tfrac{63}{5}\langle r^4\rangle^2 + 12\langle r^2\rangle\langle r^6\rangle] \\ &\quad + 42N(N-1)(N-2)\langle r^2\rangle^2\langle r^4\rangle + \tfrac{35}{3}N(N-1)(N-2)(N-3)\langle r^2\rangle^4. \end{aligned} \tag{2.16}$$

When all flights are of equal length, these expressions reduce to those listed in Flory (1969). For large N, the moments behave as

$$\langle R^{2k}|N\rangle \sim 1\cdot 3\cdot 5\ldots(2k+1)(2q^2)^k \tag{2.17}$$

with q given by equation (2.7). These asymptotic moments are those of a Maxwell PDF, equation (2.8).

The *rate* at which the moments approach their limiting values given by equation (2.17) depends upon $w(r)$. Consider the moment ratios

$$D_{2k} \equiv \frac{\langle R^{2k}|N\rangle}{1\cdot 3\ldots(2k+1)(2q^2)^k} \tag{2.18}$$

$$\sim 1+(1/N)d_{2k}+\mathrm{O}(1/N^2), \tag{2.19}$$

where d_{2k} depends upon $w(r)$. Note that $D_2 \equiv 1$ for all N. For flights of equal fixed length, we have

$$d_4 = -\tfrac{2}{5}, \qquad d_6 = -\tfrac{6}{5}, \qquad d_8 = -\tfrac{12}{5}, \tag{2.20}$$

while for flights obeying a rectangular density function

$$d_4 = \tfrac{2}{25}, \qquad d_6 = \tfrac{6}{25}, \qquad d_8 = \tfrac{12}{25}. \tag{2.21}$$

The values of the d's are negative for the fixed length case and five times larger in magnitude than the corresponding d's for the rectangular density case. Thus, as we would expect, the rectangular density moments approach their limiting values more rapidly than do the fixed length moments. A similar phenomenon occurs in the two-dimensional situation (Barakat and Cole 1979).

3. Random N: Poisson distribution

We now consider the situation where in addition N is also allowed to be a random variable governed by a probability distribution $P(N)$. By elementary probability theory, the characteristic function $A(\rho)$ and radial density function $f(R)$ are given by

$$A(\rho) = \sum_{N=0}^{\infty} A(\rho|N)P(N), \tag{3.1}$$

$$f(R) = \sum_{N=0}^{\infty} f(R|N)P(N). \tag{3.2}$$

In this section, $P(N)$ is taken to be a Poisson distribution

$$P(N) = (1/N!)\langle N\rangle^N \mathrm{e}^{-\langle N\rangle}, \tag{3.3}$$

where $\langle N\rangle$ is the average of N with respect to the Poisson distribution. We alter the notation and write $A(\rho)=A(\rho|\langle N\rangle)$, $f(R)=f(R|\langle N\rangle)$ to denote the explicit dependence on $\langle N\rangle$.

The characteristic function can be written in closed form

$$A(\rho|\langle N\rangle) = \sum_{N=0}^{\infty} [A(\rho|1)]^N \frac{\langle N\rangle^N \mathrm{e}^{-\langle N\rangle}}{N!} = \exp\{-\langle N\rangle[1-A(\rho|1)]\}. \tag{3.4}$$

The corresponding radial density function can be expressed, using equation (2.5), in the form

$$f(R|\langle N\rangle)=\frac{2R}{\pi}\int_0^{\infty} A(\rho|\langle N\rangle)\rho \sin R\rho \, d\rho. \tag{3.5}$$

Now $f(R|\langle N\rangle)$ is a *mixed* density function because it contains impulse (Dirac delta function) components in addition to the absolutely continuous components. This is easily seen by direct examination of equation (3.2) and noting that

$$f(R|0)=\delta(R). \tag{3.6}$$

In the special case where all flights are of fixed length l,

$$f(R|1)=\delta(R-l). \tag{3.7}$$

However, $f(R|N)$ is absolutely continuous for $N\geqslant 2$. As $\langle N\rangle$ increases, the impulse terms tend to zero. Figures 1 and 2 show the behaviour of the absolutely continuous component of the radial density function $f(R|\langle N\rangle)$ for $\langle N\rangle=1$, 3, 4 and flights of equal deterministic length. When the flights are governed by a rectangular density only $f(R|0)$ contributes a Dirac delta function. Furthermore, the radial density functions $f(R|N)$ are practically Maxwellian for $N\geqslant 3$, see figures 3 and 4 of Barakat

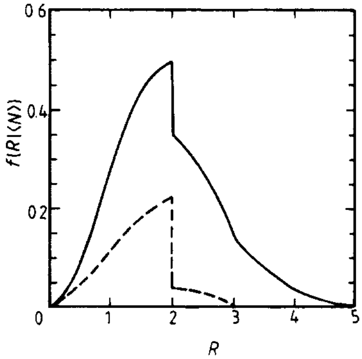

Figure 1. Absolutely continuous component of radial density function for flights of equal length: - - - $\langle N\rangle=1$; —— $\langle N\rangle=4$.

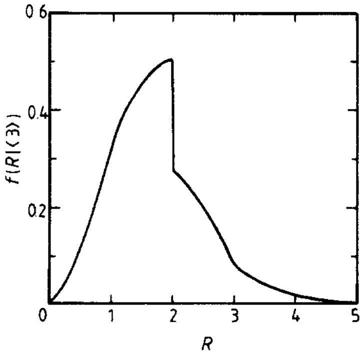

Figure 2. Absolutely continuous component of radial density function for flights of equal length when $\langle N\rangle=3$.

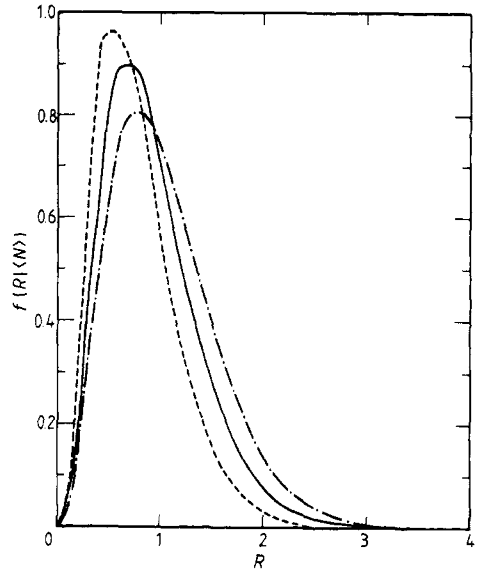

Figure 3. Absolutely continuous component of radial density function when individual flights are governed by rectangular density functions: - - - $\langle N\rangle = 2$, —— $\langle N\rangle = 3$, - · - · - $\langle N\rangle = 4$.

(1973). Consequently, the absolutely continuous component of $F(R|\langle N\rangle)$ will be very smooth, even for small $\langle N\rangle$. This is borne out by the numerical calculations displayed in figure 3.

When $\langle N\rangle$ is large, the characteristic function $A(\rho|\langle N\rangle)$ behaves as a Gaussian in the vicinity of the origin,

$$A(\rho|\langle N\rangle) \sim \exp(-q_1^2\rho^2), \tag{3.8}$$

where

$$q_1 \equiv (\langle N\rangle\langle r^2\rangle/6)^{1/2}. \tag{3.9}$$

In analogy with equation (2.8), $f(R|\langle N\rangle)$ is also Maxwellian. This is to be expected; as $\langle N\rangle$ is increased, the Poisson distribution becomes very peaked around $\langle N\rangle = N$ and behaves somewhat like a Dirac delta function centred at $\langle N\rangle = N$. Consequently, the dominant term in the series, equation (3.2), is the term $\langle N\rangle = N$.

The corresponding moments $\langle R^{2k}|\langle N\rangle\rangle$ are easily evaluated. They are given by

$$\langle R^{2k}|\langle N\rangle\rangle = \sum_{N=0}^{\infty} \langle R^{2k}|N\rangle P(N). \tag{3.10}$$

Substitution of the appropriate $\langle R^{2k}|N\rangle$ from equation (2.16) and subsequent summation of the series yields the desired expressions. The first four are:

$$\begin{aligned}
\langle R^2|\langle N\rangle\rangle &= \langle N\rangle\langle r^2\rangle,\\
\langle R^4|\langle N\rangle\rangle &= \langle N\rangle\langle r^4\rangle + \tfrac{5}{3}\langle N\rangle^2\langle r^2\rangle^2,\\
\langle R^6|\langle N\rangle\rangle &= \langle N\rangle\langle r^6\rangle + 7\langle N\rangle^2\langle r^2\rangle\langle r^4\rangle + \tfrac{35}{9}\langle N\rangle^3\langle r^2\rangle^3,\\
\langle R^8|\langle N\rangle\rangle &= \langle N\rangle\langle r^8\rangle + 12\langle N\rangle^2\langle r^2\rangle\langle r^6\rangle + \tfrac{35}{3}\langle N\rangle^4\langle r^2\rangle^4\\
&\quad + \tfrac{63}{5}\langle N\rangle^2\langle r^4\rangle^2 + 42\langle N\rangle^3\langle r^2\rangle^2\langle r^4\rangle.
\end{aligned} \tag{3.11}$$

The large $\langle N\rangle$ limit of these expressions is the same as for the large N limit, equation (2.17), with $N\to\langle N\rangle$.

The rate of approach of these moments to their limiting values will be characterised by the moment ratios S_{2k} analogous to D_{2k},

$$S_{2k}\equiv\frac{\langle R^{2k}|\langle N\rangle\rangle}{1\cdot 3\dots(2k+1)(2q_1^2)^k}\sim 1+\frac{1}{\langle N\rangle}s_{2k}+\mathrm{O}\left(\frac{1}{\langle N\rangle^2}\right). \tag{3.12}$$

The s_{2k} also depend upon $w(r)$. $S_2\equiv 1$ for all $\langle N\rangle$. For flights of fixed length

$$s_4=\tfrac{3}{5},\qquad s_6=\tfrac{9}{5},\qquad s_8=\tfrac{18}{5}, \tag{3.13}$$

while for flights obeying a rectangular density

$$s_4=\tfrac{27}{25},\qquad s_6=\tfrac{81}{25},\qquad s_8=\tfrac{162}{25}. \tag{3.14}$$

The s_{2k} are positive in both cases, thus $S_{2k}\geqslant 1$. Because the rectangular density values of the s_{2k} are $\frac{9}{5}$ as large as those for the fixed length case, we now have the opposite situation to that for the deterministic N case, namely that the rectangular density now has a slower rate of approach to the limiting value than the fixed length case.

4. Random N: negative binomial distribution

We next examine the case where N is governed by a negative binomial distribution function. In the two-dimensional random walk, this problem has been studied by Jakeman and associates; a summary of their work is given in Jakeman (1980a). When the average number of flights is large, they show that the limiting radial density function is governed by a modified Bessel function of the second kind which they term a K-density function (see also Siddiqui and Weiss 1963).

The negative binomial distribution referred to its mean $\langle N\rangle$ is given by

$$P(N)=\binom{N+\alpha-1}{N}\frac{(\langle N\rangle/\alpha)^N}{(1+\langle N\rangle/\alpha)^{N+\alpha}} \tag{4.1}$$

where α is real and $\geqslant 1$. As in the Poisson case, $f(R|\langle N\rangle)$ is also a mixed density function containing an impulse component at the origin. There is an additional impulse component at $R=l$ when all flights are of equal deterministic length l. The strengths of these impulse components tend to zero as $\langle N\rangle$ increases although at a slower rate than for the Poisson case. For example, the $N=0$ component for the Poisson case at $\langle N\rangle=4$ is 0.0183, while the corresponding component for the negative binomial with $\alpha=2$ is 0.1111 and with $\alpha=10$ is 0.0345.

The characteristic function $A(\rho|\langle N\rangle)$ can be evaluated in closed form in much the same manner as in § 3. The final result is

$$A(\rho|\langle N\rangle)=\{1+(\langle N\rangle/\alpha)[1-A(\rho|1)]\}^{-\alpha}. \tag{4.2}$$

For fixed $\langle N\rangle$, and increasing α, we have

$$A(\rho|\langle N\rangle)\sim\exp\{-\langle N\rangle[1-A(\rho|1)]\} \tag{4.3}$$

upon using the fact that for large α

$$(1+b/\alpha)^{-\alpha}\sim \mathrm{e}^{-b}(1-b^2/2\alpha+\dots). \tag{4.4}$$

Thus, when α is very large, the characteristic function $A(\rho|\langle N\rangle)$ and density function $f(R|\langle N\rangle)$ approximate those in § 3 devoted to the Poisson distribution.

For arbitrary α and large $\langle N\rangle$, the characteristic function behaves as

$$A(\rho|\langle N\rangle)\sim(1+q_2^2\rho^2)^{-\alpha} \tag{4.5}$$

in the vicinity of the origin, where

$$q_2\equiv(\langle N\rangle\langle r^2\rangle/6\alpha)^{1/2}. \tag{4.6}$$

Note that q_2 now depends upon the parameter α. The corresponding radial density function is

$$f(R|\langle N\rangle)=\frac{2R}{\pi q_2^2}\int_0^\infty x(1+x^2)^{-\alpha}\sin\left(\frac{Rx}{q_2}\right)\mathrm{d}x. \tag{4.7}$$

The integral can be evaluated by differentiation of the known integral (Watson 1944)

$$\int_0^\infty(1+x^2)^{-\alpha}\cos yx\,\mathrm{d}x=\frac{\pi^{1/2}}{\Gamma(\alpha)}\left(\frac{y}{2}\right)^{\alpha-1/2}K_{\alpha-1/2}(y), \tag{4.8}$$

where K is the modified Bessel function of the second kind, and the reduction formula

$$\frac{\mathrm{d}}{\mathrm{d}y}[y^\nu K_\nu(y)]=-y^\nu K_{\nu-1}(y). \tag{4.9}$$

The final result is

$$f(R|\langle N\rangle)=\frac{4(R/2q_2)^{\alpha+1/2}}{\pi^{1/2}\Gamma(\alpha)q_2}K_{\alpha-3/2}(R/q_2) \tag{4.10}$$

which is the random flight version of the K-density function.

The moments about the origin, $\langle R^{2k}|\langle N\rangle\rangle$, as evaluated by direct summation

$$\langle R^{2k}|\langle N\rangle\rangle=\sum_{N=0}^\infty\langle R^{2k}|N\rangle\binom{N+\alpha-1}{N}\frac{(\langle N\rangle/\alpha)^N}{(1+\langle N\rangle/\alpha)^{N+\alpha}}, \tag{4.11}$$

yield for the first four moments:

$$\begin{aligned}
\langle R^2|\langle N\rangle\rangle&=\langle N\rangle\langle r^2\rangle,\\
\langle R^4|\langle N\rangle\rangle&=\langle N\rangle\langle r^4\rangle+\frac{5(\alpha+1)}{3\alpha}\langle N\rangle^2\langle r^2\rangle^2,\\
\langle R^6|\langle N\rangle\rangle&=\langle N\rangle\langle r^6\rangle+\frac{7(\alpha+1)}{\alpha}\langle N\rangle^2\langle r^2\rangle\langle r^4\rangle+\frac{35(\alpha+1)(\alpha+2)}{9\alpha^2}\langle N\rangle^3\langle r^2\rangle^3,\\
\langle R^8|\langle N\rangle\rangle&=\langle N\rangle\langle r^8\rangle+\frac{12(\alpha+1)}{\alpha}\langle N\rangle^2\langle r^2\rangle\langle r^6\rangle+\frac{63(\alpha+1)}{5\alpha}\langle N\rangle^2\langle r^4\rangle^2\\
&\quad+\frac{42(\alpha+1)(\alpha+2)}{\alpha^2}\langle N\rangle^3\langle r^2\rangle^2\langle r^4\rangle+\frac{35(\alpha+1)(\alpha+2)(\alpha+3)}{3\alpha^3}\langle N\rangle^4\langle r^2\rangle^4.
\end{aligned} \tag{4.12}$$

When $\langle N\rangle$ is very large, the dominant term is

$$\langle R^{2k}|\langle N\rangle\rangle\sim[1\cdot3\cdot5\ldots(2k+1)(2q^2)^k]\frac{(\alpha+1)(\alpha+2)\ldots(\alpha+k-1)}{\alpha^{k-1}}. \tag{4.13}$$

The term in square brackets is the term characteristic of a Maxwell radial density function, see equation (2.17). However, it is modulated by a term depending upon the negative binomial parameter α; this term tends to unity as α is made large.

5. Comments

The limiting form of the radial density function, equation (2.8), as deterministic N is made very large belongs to the class of infinitely divisible density functions. A density function is infinitely divisible if its characteristic function is the Nth power of some characteristic function (Gnedenko and Kolmogorov 1964, Petrov 1975). The characteristic function, equation (2.6), corresponding to the limiting radial density function, equation (2.8), can be written

$$A(\rho|N)=[\exp(-\tfrac{1}{6}\langle r^2\rangle\rho^2)]^N=[A(\rho|1)]^N. \quad (5.1)$$

The term in square brackets is itself a characteristic function implying that the Maxwellian radial density, equation (2.8), is infinitely divisible. The infinite divisibility property translates into the condition that the density functions retain their *functional* forms irrespective of the statistical properties of the individual random flights.

An important subclass of infinitely divisible densities are the stable density functions. Unlike the infinitely divisible class, for which the distributions can be absolutely continuous (i.e. have a density function) or discrete, the stable subclass can only be absolutely continuous. When expressed in terms of the corresponding characteristic functions, the stable requirement is: a density function is stable if for every real a_1, $a_2>0$ there exist real numbers $a>0$, b such that the characteristic functions obey

$$A(a_1\rho)A(a_2\rho)=e^{ib\rho}A(a\rho). \quad (5.2)$$

Obviously the limiting characteristic function, equation (2.6), satisfies this equation with $a^2=a_1^2+a_2^2$, $b\equiv 0$. Equation (5.2) is equivalent to the statement that the convolution of any two radial density functions of the same type is also a *rescaled* version of the same density function. Consequently we can take as a universal independent variable $R'\equiv R/2q$ in the Maxwellian radial density, equation (2.8).

In the mathematical literature (see Gnedenko and Kolmogorov 1964, Petrov 1975) it is shown that the class of limit laws for sums of independent random variables as deterministic N becomes very large coincides with the class of infinitely divisible density functions. Jona-Lasinio (1975) connected the stable subclass of infinitely divisible distributions with the renormalisation group approach in statistical mechanics. Stable distributions also play a prominent role in Mandelbrot's (1977) theory of fractals.

When we allow the number of flights N to be random, the limiting situation (in the sense that $\langle N\rangle\to\infty$) is more complicated because the theory of such limit laws only holds for deterministic N. We must expect that the limiting radial density functions will depend upon the structure of the probability distribution governing N. When N is governed by the Poisson distribution or the negative binomial distribution, the limiting radial densities are infinitely divisible, as the reader can easily verify from the corresponding characteristic functions, equations (3.8) and (4.5). We note that the Poisson and negative binomial distributions are themselves infinitely divisible. Thus if N is governed by these two distributions the limiting radial densities retain their functional form.

The limiting radial density for Poisson distributed N is also Maxwellian, as noted in § 3; explicitly

$$f(R|\langle N\rangle) = (2\pi)^{-1/2} q_1^{-3} R^2 \exp(-R^2/4q_1^2). \tag{5.3}$$

Consequently this limiting radial density is also stable. This result is intuitively obvious; when $\langle N\rangle$ is very large, the Poisson distribution acts like a Dirac delta function centred about $\langle N\rangle = N$. Consequently the dominant term in equation (3.2) is $\langle N\rangle = N$ and we have the usual argument leading to a stable density. The universal independent variable in this case is $R'' \equiv R/2q_1$.

The limiting K-density function, equation (4.10), is *not* stable. From a formal viewpoint, this is obvious because there does not exist a real $a > 0$ such that

$$(1+q_2^2 a_1^2 \rho^2)^{-\alpha}(1+q_2^2 a_2^2 \rho^2)^{-\alpha} = (1+q_2^2 a^2 \rho^2)^{-\alpha}. \tag{5.4}$$

The K-density function is governed by two parameters α and q_2 (with q_2 itself a function of α). No amount of rescaling can reduce the dependence to just one parameter as with the stable Maxwellian density functions. Furthermore, the order of the modified Bessel function depends upon α. Although the functional dependence is unaltered, the numerical shapes assumed by the K-density depend upon α and there is no universal independent variable!

It would be of some interest to know what subclass of infinitely divisible $P(N)$ leads to stable density functions.

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